Exact Baryon, Strangeness and Charge Conservation

in Hadronic Gas Models.

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Relativistic heavy ion collisions are studied assuming that particles can be described by a hadron gas in thermal and chemical equilibrium. The exact conservation of baryon number, strangeness and charge are explicitly taken into account. For heavy ions the effect arising from the neutron surplus becomes important and leads to a substantial increase in e.g. the π^-/π^+ ratio. A method is developed which is very well suited for the study of small systems.

I. INTRODUCTION

It has been shown recently in a very convincing manner that hadronic matter produced in collisions of small systems (e^+e^-, $p-p$, $p-\overline{p}$) is very close to chemical equilibrium. With only a small number of parameters, namely the temperature T, the volume V and, a factor γ_S which measures the deviation of strange particles from chemical equilibrium, it is possible to fit close to 30 different particle abundances [1,2]. It is of great interest to establish whether or not this also holds for systems of medium size like $S-S$ and for relativistic heavy ion collisions like $Au-Au$ at the BNL-AGS or $Pb-Pb$ at CERN [3]. This is of relevance because it is widely believed that in these collisions one could create a quark-gluon plasma. Of immediate interest is the observed increase in the abundance of strange particles in heavy ion collisions. In this paper we would like to present a method for taking into account the exact conservation of baryon number B, strangeness S and charge Q. The method is ideally suited to analyze the extension from small systems like $p-p$ towards bigger systems up to $S-S$. For larger systems the method becomes impractical and should be replaced by a more straightforward approach based on the grand canonical ensemble. Our numerical results can be compared with data from the Brookhaven National Laboratory, e.g. the E866 collaboration [4] which measures the dependence of hadronic ratios on the number of projectile participants thereby allowing one to study the transition from small to large systems in a systematic way. These results give insight into the behavior of the produced hadronic system as a function its size. The treatment presented in this paper differs from the standard one based on the grand canonical ensemble (see for example [5–7] in that we consider the quantum number content exactly. This means that we do not introduce chemical potentials for the baryon number or for strangeness. Chemical potentials are usually introduced to enforce the right quantum numbers of the system in an average sense. This is a correct treatment for large systems, however, for small systems the production of extra proton - anti-proton pairs for example will clearly be more suppressed than in large systems. These extra corrections were first pointed out by Hagedorn [8] and subsequently a complete treatment has been formulated [9–16]. We emphasize that these corrections do not contain information about the dynamics. They simply follow from baryon number conservation and must be taken into account before considering more involved models. It is also worth emphasizing that they do not introduce any new parameters. Our treatment differs from the one presented in [1,2] in that it is much more analytic. It extends the analysis started in [17] since it also includes the exact treatment of charge conservation and allows for the correct treatment of systems which are not isospin-symmetric.

II. PARTITION FUNCTION

The exact treatment of quantum numbers in statistical mechanics has been well established for some time [9]. It is obtained by projecting the partition function onto the desired values of B, Q and S

$$Z_{B,Q,S} = \frac{1}{2\pi} \int_0^{2\pi} d\phi \; e^{-iB\phi} \; \frac{1}{2\pi} \int_0^{2\pi} d\alpha \; e^{-iQ\alpha} \frac{1}{2\pi} \int_0^{2\pi} d\psi e^{-iS\psi} Z(T, \lambda_B, \lambda_Q, \lambda_S) \tag{2.1}$$

where Z is the (grand canonical) partition function and the usual fugacity factors λ_B and λ_S have been replaced by :

$$\lambda_B = e^{i\phi} \qquad \lambda_Q = e^{i\alpha} \qquad \lambda_S = e^{i\psi}. \tag{2.2}$$

As the contributions always come pairwise for particle and anti-particle, the fugacity factors will give rise to the cosine of the angle. In the extended treatment it is useful to group all particles appearing in the Particle Data Booklet [18] into fourteen categories depending on their quantum numbers (we leave out charm and bottom). Z_K is the sum (given below) of all mesons having strangeness ± 1 $(K, \bar{K}, K^*, \ldots)$ and zero charge. Similarly Z_N is the sum of all baryons and anti-baryons having zero strangeness, Z_Y is the sum of all hyperons and anti-hyperons, while Z_0 is the sum of all non-strange mesons.

Quantum Numbers	Lowest Mass Particle	Notation
$S = 0$ $B = 0$ $Q = 0$	π^0	Z_0
$S = 0$ $B = 0$ $Q = 1$	π^+	Z_{π^c}
$S = 0$ $B = 1$ $Q = -1$	Δ^-	Z_{Δ^-}
$S = 0$ $B = 1$ $Q = 0$	n	Z_n
$S = 0$ $B = 1$ $Q = 1$	p	Z_p
$S = 0$ $B = 1$ $Q = 2$	Δ^{++}	$Z_{\Delta^{++}}$
$S = 1$ $B = 0$ $Q = 0$	K^0	Z_K
$S = 1$ $B = 0$ $Q = 1$	K^+	Z_{K^c}
$S = -1$ $B = 1$ $Q = 1$	Σ^+	Z_{Σ^+}
$S = -1$ $B = 1$ $Q = -1$	Σ^-	Z_{Σ^-}
$S = -1$ $B = 1$ $Q = 0$	Λ	Z_Λ
$S = -2$ $B = 1$ $Q = 0$	Ξ^0	Z_{Ξ^0}
$S = -2$ $B = 1$ $Q = -1$	Ξ^-	Z_{Ξ^-}
$S = -3$ $B = 1$ $Q = -1$	Ω	Z_Ω

Table 1 : Classification of the families of particles.

For example, Z_{K^c} is the single particle partition function of all charged strange mesons $(K^+, K^-, K^{*+}, K^{*-}...)$ i.e.

$$Z_{K^c} = \sum_{j \in |S|=1, |B|=0, |Q|=1} g_j V \int \frac{d^3 p}{(2\pi)^3} e^{-E_j/T}, \tag{2.3}$$

where g_j is the degeneracy factor.

With the help of the definitions presented in table 1, the partition function can be rewritten in the following form,

$$\begin{aligned}
Z_{B,S,Q}(T,V) = {} & \frac{1}{(2\pi)^3} \int_0^{2\pi} d\phi \, \exp(-iS\phi) \int_0^{2\pi} d\psi \, \exp(-iB\psi) \int_0^{2\pi} d\alpha \, \exp(-iQ\alpha) \\
& \cdot \exp[2Z_K \cos\phi + 2Z_n \cos\psi + 2Z_{\pi^c} \cos\alpha + 2Z_\Lambda \cos(\psi - \phi) \\
& + 2Z_{K^c} \cos(\phi + \alpha) + 2Z_{\Delta^-} \cos(\psi - \alpha) + 2Z_p \cos(\psi + \alpha) + 2Z_{\Delta^{++}} \cos(\psi + 2\alpha) \\
& + 2Z_{\Sigma^+} \cos(-\phi + \psi + \alpha) + 2Z_{\Sigma^-} \cos(-\phi + \psi - \alpha) \\
& + 2Z_{\Xi^-} \cos(-2\phi + \psi - \alpha) + 2Z_{\Xi^0} \cos(-2\phi + \psi) \\
& + 2Z_\Omega \cos(-3\phi + \psi - \alpha)] \cdot \exp(Z_0).
\end{aligned} \tag{2.4}$$

To continue we would like to make use of the integral representation of the modified Bessel function

$$I_n(z) = \frac{1}{\pi} \int_0^\pi e^{z\cos\theta} \cos n\theta. d\theta \tag{2.5}$$

However, this cannot be used in a straightforward manner since the dependence on angles does not factorize and almost all terms involve more than one angle. To circumvent this difficulty we introduce new angles whenever more than one appears. For example, for the term involving Z_Λ we introduce an intermediate angle λ in the following way

$$1 = \int_0^{2\pi} d\lambda \ \delta(\psi - \phi - \lambda)$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{\pi} d\lambda \ e^{in(\psi - \phi - \lambda)} \tag{2.6}$$

With this method complete factorization can be achieved at the price of introducing a new summation at each step. Since there are ten cosines in Eq. (2.4) involving more than one angle, we end up having ten summations. The following result is obtained [1]

$$Z_{B,S,Q}(T,V) = \left[\prod_{j=1}^{10} \sum_{n_j=-\infty}^{\infty}\right] I_{\nu_S}\left(2Z^{K^0}\right) I_{\nu_B}(2Z^n) I_{\nu_Q}(2Z_{\pi^c})$$
$$I_{-n_1}(2Z_{\Lambda^0}) I_{-n_2}(2Z_{K^c}) I_{-n_3}(2Z_p)$$
$$I_{-n_4}(2Z_{\Delta^-}) I_{-n_5}(2Z_{\Sigma^+}) I_{-n_6}(2Z_{\Sigma^-}) I_{-n_7}(2Z_{\Xi^0})$$
$$I_{-n_8}(2Z_{\Xi^-}) I_{-n_9}(2Z_{\Omega}) I_{-n_{10}}(2Z_{\Delta^{++}}) \cdot \exp(Z_0) \tag{2.7}$$

where

$$\nu_S = S - n_1 + n_2 - n_5 - n_6 - 2n_7 - 2n_8 - 3n_9 \tag{2.8}$$
$$\nu_B = B + n_1 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8 + n_9 + n_{10} \tag{2.9}$$
$$\nu_Q = Q + n_2 + n_3 - n_4 + n_5 - n_6 - n_8 - n_9 + 2n_{10}. \tag{2.10}$$

Equation (2.7) is our main result and forms the basis of our analysis below. The sums in the above equation are dominated by terms corresponding to the index of the Bessel functions being zero (for the size of the system being sufficiently small). This means however that for large values of the baryon number B some of the indices could become quite large. In such cases evaluation of the sums becomes very time-consuming numerically and it is better to resort to the grand canonical ensemble.

The canonical partition function for a gas with three conserved quantum numbers has thus been derived and will be used in the following to derive expressions for particle numbers. The differentiation of the equation (2.7) for particle abundances decreases/increases some of the indices ν_S, ν_B and ν_Q by one and leads to useful R-factors which, for different particle species deviate from $Z_{B,S,Q}(T,V)$ in the way indicated below

$$
\begin{array}{llll}
R_{K^+/K^-}: & & \nu_S \to \nu_S \mp 1; & \nu_Q \to \nu_Q \mp 1 \\
R_{K^0/\overline{K^0}}: & & \nu_S \to \nu_S \mp 1 \\
R_{p/\bar{p}}: & \nu_B \to \nu_B \mp 1; & & \nu_Q \to \nu_Q \mp 1 \\
R_{\Lambda/\bar{\Lambda}}: & \nu_B \to \nu_B \mp 1; & \nu_S \to \nu_S \pm 1 \\
R_{\Sigma^+/\bar{\Sigma}^+}: & \nu_B \to \nu_B \mp 1; & \nu_S \to \nu_S \mp 1; & \nu_Q \to \nu_Q \mp 1
\end{array}
\tag{2.11}
$$

If a particle, i, has strangeness 1, baryon number 0 and charge 0, it's density will be given by

$$n_i = \left[\frac{R_K}{Z_{B,S,Q}}\right] g_i \int \frac{d^3p}{(2\pi)^3} e^{-E_i/T}, \tag{2.12}$$

with

$$R_K = \left[\prod_{j=1}^{10} \sum_{n_j=-\infty}^{\infty}\right] I_{\nu_S-1}\left(2Z^{K^0}\right) I_{\nu_B}(2Z^n) I_{\nu_Q}(2Z_{\pi^c})$$
$$I_{-n_1}(2Z_{\Lambda^0}) I_{-n_2}(2Z_{K^c}) I_{-n_3}(2Z_p)$$
$$I_{-n_4}(2Z_{\Delta^-}) I_{-n_5}(2Z_{\Sigma^+}) I_{-n_6}(2Z_{\Sigma^-}) I_{-n_7}(2Z_{\Xi^0})$$
$$I_{-n_8}(2Z_{\Xi^-}) I_{-n_9}(2Z_{\Omega}) I_{-n_{10}}(2Z_{\Delta^{++}}) \cdot \exp(Z_0). \tag{2.13}$$

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[1] Full details can be found in [19].

All other particle densities are obtained by using the appropriate R factor. The factor in square brackets in equations (2.12) and (2.13) replaces the fugacity in the usual grand canonical ensemble treatment [5–7]. Having thus determined all particle densities, we consider the behavior at freeze-out time. In this case all the resonances in the gas are allowed to decay into lighter stable particles. This means that each particle density is multiplied with its appropriate branching ratio (indicated by Br below). The abundances of particles in the final state are thus determined by :

$$n_H = \sum_i n_i Br(i \to H) \tag{2.14}$$

where each sum runs over all particles contained in the hadronic gas and H refers to a hadron ($\pi^+, K^+, \ldots$).

III. NUMERICAL RESULTS

In order to stay close to the grand canonical ensemble treatment we keep the temperature and the density B/V fixed. The latter corresponds to keeping the baryon chemical potential fixed. We therefore expect all ratios to become constant as the size of the system is increased. Fig. 1 illustrates the approach to the thermodynamic (grand canonical) limit for the K^+/π^+ ratio. As the system becomes larger the ratio approaches a constant. The variation of the π^-/π^+ ratio as one increases the neutron surplus or, equivalently, the ratio $B/2Q$, is shown in Fig. 2. As expected one observes a clear increase of this ratio with $B/2Q$. In Fig. 3 we show the p/π^+ ratio as a function of the baryon number B. The K^+/K^- ratio is shown in Fig. 4 for a fixed value of the temperature, $T = 110$ MeV. One observes a very smooth decrease and this ratio very quickly approaches its asymptotic value.

Fig. 5 shows the $\bar{p}/\pi^+$ ratio as a function of the size of the system. This figure is interesting because preliminary results from the E866 collaboration [4] show a *decrease* of this ratio with increasing system size, i.e. the opposite behavior from the one observed in Fig. 5. In our opinion, this proves in a convincing manner that $\bar{p}$ are definitely not in thermal equilibrium. It would be difficult to show this if the analysis is based on only one system.

IV. SUMMARY

We have assumed that, as a first approximation, the hadronic final state produced in a relativistic heavy ion collision can be described by a hadronic gas which is in chemical equilibrium. The standard description of such a system makes use of a statistical description based on the grand canonical ensemble using chemical potentials for the conserved quantum numbers. However, if the system is very small then there will *always* be corrections due to the size of the system. We have presented a method for taking into account corrections arising from the exact conservation of baryon number B, strangeness S and, charge Q. This method is well suited for small systems and numerical results have been presented for the most abundantly observed hadrons. For large systems the numerical evaluation becomes very slow because a large number of terms have to be kept in the sums. The numerical results presented can be compared with data from Brookhaven, e.g. the E866 collaboration [4]. We plan to do this in the near future.

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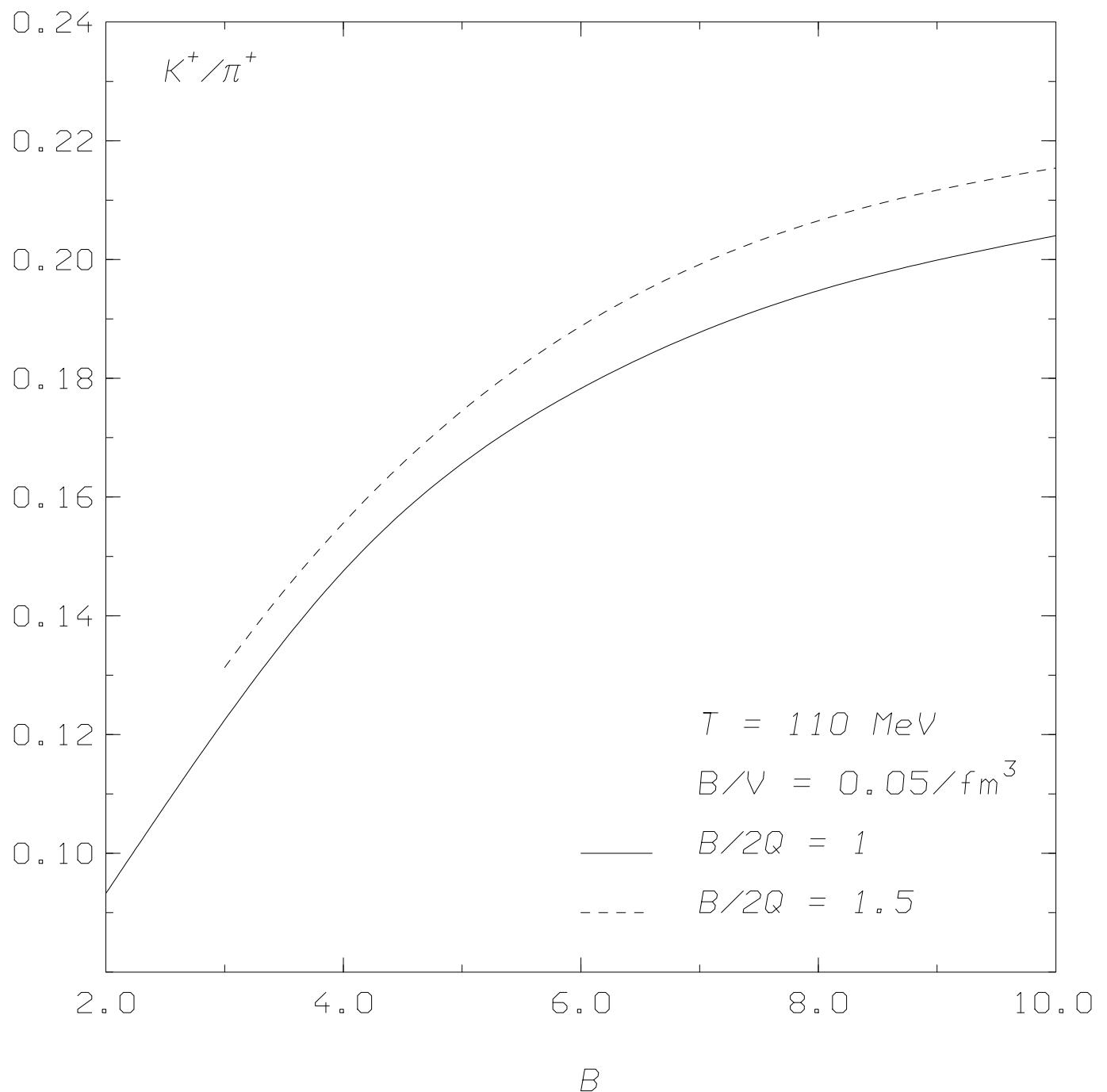[19] M. Marais, M.Sc. thesis, University of Cape Town, March 1977.

FIG. 1. The K^+/π^+ ratio as a function of the baryon number B for fixed values of the temperature T, the baryon density B/V and $B/2Q$.
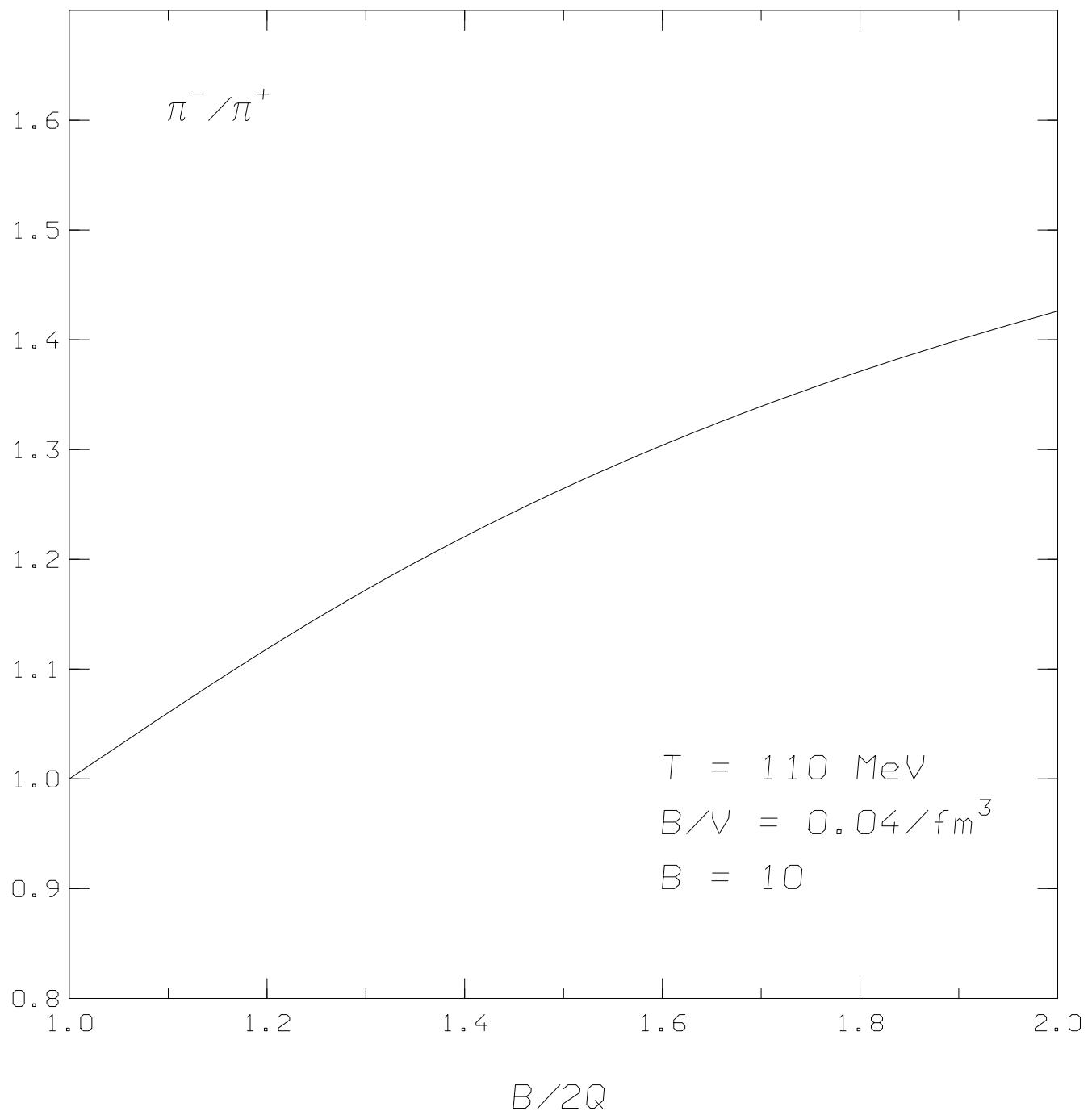
FIG. 2. The π^-/π^+ ratio as a function of the $B/2Q$ for a fixed value of the temperature T, the baryon density B/V, and the baryon number ($B = 10$).
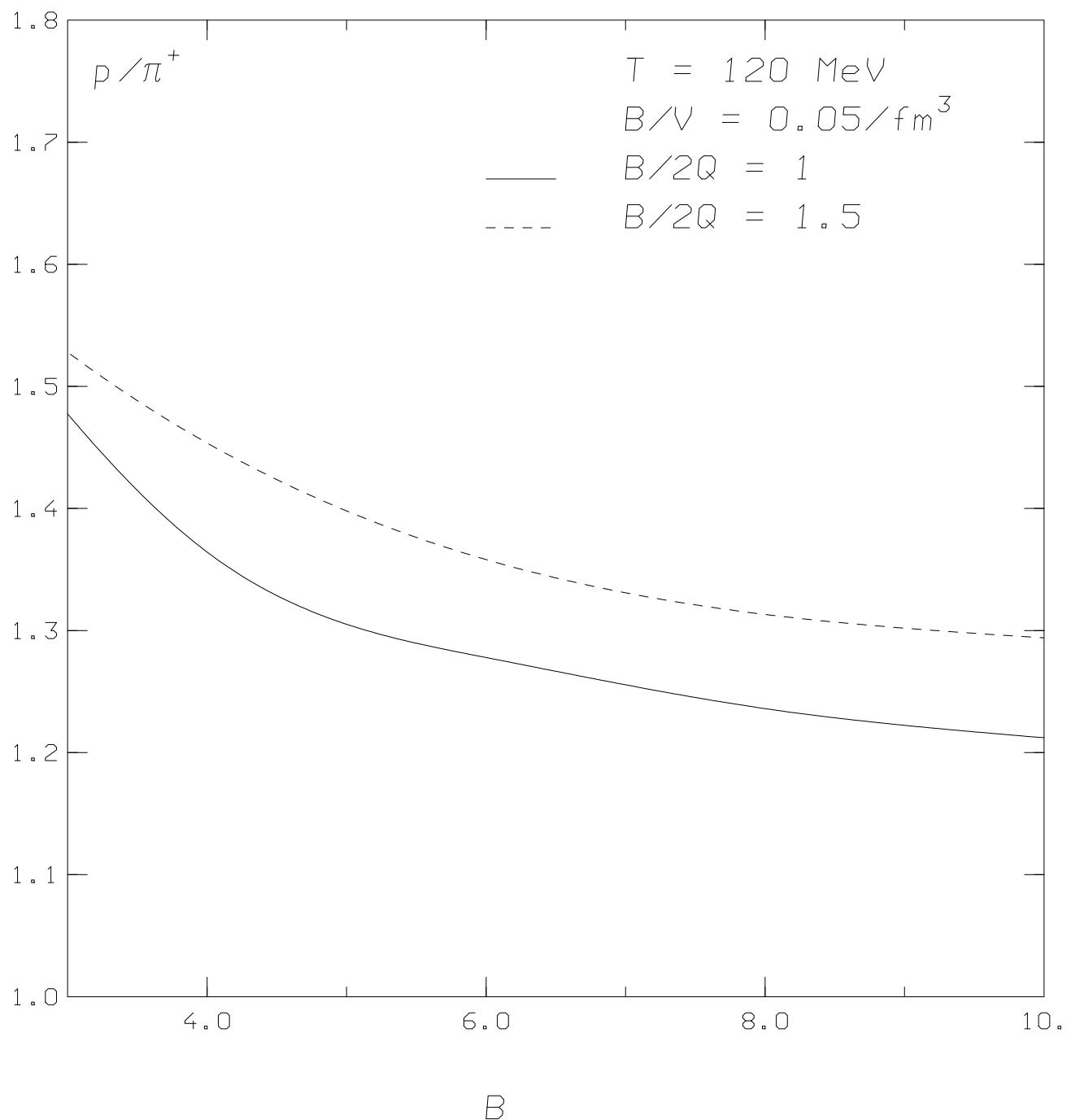
FIG. 3. The p/π^+ ratio as a function of the baryon number B for fixed values of the temperature T, the baryon density B/V and $B/2Q$.

FIG. 4. The K^+/K^- ratio as a function of the the baryon number B for fixed values of the temperature T, the baryon density B/V and $B/2Q$.
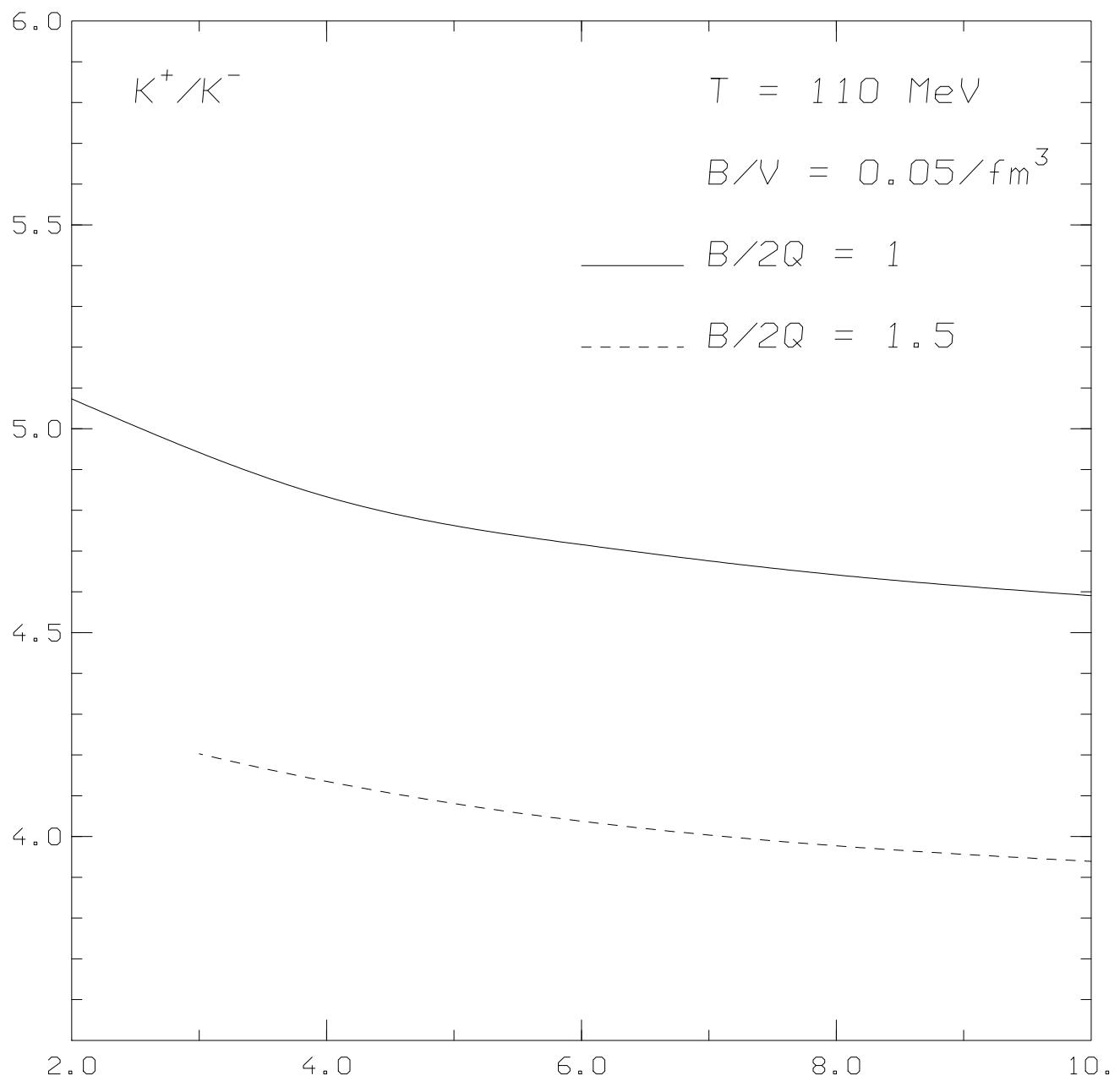
FIG. 5. The $\bar{p}/\pi^+$ ratio as a function of the baryon number B for fixed values of the temperature T, the baryon density B/V and $B/2Q$.

π^-/π^+

$T = 110$ MeV

$B/V = 0.04/fm^3$

$B = 10$

$B/2Q$

$\bar{p}/\pi^{+}$

$T = 110$ MeV

$B/V = 0.05/fm^3$

—— $B/2Q = 1.0$

- - - $B/2Q = 1.5$

0.0002

0.0001

2.0   4.0   6.0 B 8.0   10.